

$N_f = 2$ linear sigma model in presence of axial anomaly from Functional Renormalization Group

Mara Grahl

my supervisors:

Francesco Giacosa and Dirk H. Rischke

Institut für Theoretische Physik



Brookhaven National Laboratory, New York
May 31, 2012

Outline of the talk

- 1 The $O(4)$ -conjecture
 - The universality hypothesis...
 - ...applied to two-flavor QCD
- 2 Functional Renormalization Group (FRG) Method
 - Local Potential Approximation (LPA)
- 3 Two-flavor linear sigma model in absence of anomaly, LPA
- 4 Infrared fixed points in presence of the anomaly
- 5 Conclusion

The universality hypothesis

Experimental facts for (simple) systems:
power law behavior near critical point

- order parameter $\propto |T - T_c|^\beta$
- specific heat: $C \propto |T - T_c|^\alpha$
- susceptibility: $\chi \propto |T - T_c|^{-\gamma}$
- correlation length: $\xi \propto |T - T_c|^{-\nu}$
- etc.

scaling relations

- $\alpha = 2 - \nu D$
- $\beta = \frac{\nu}{2} (D - 2 + \eta)$
- $\gamma = \nu (2 - \eta)$
- etc.

The universality hypothesis

Universality hypothesis (R.B.Griffiths, 1970) The critical exponents (primarily) depend on three properties:

- ① the lattice dimensionality (i.e. the spatial dimensionality D of the system)
- ② the spin dimensionality (i.e. the number of components of the order parameter)
- ③ the range of interaction between the spins on the lattice

The universality hypothesis

Universality hypothesis (Bruce, 1980) The critical exponents depend on at least four properties:

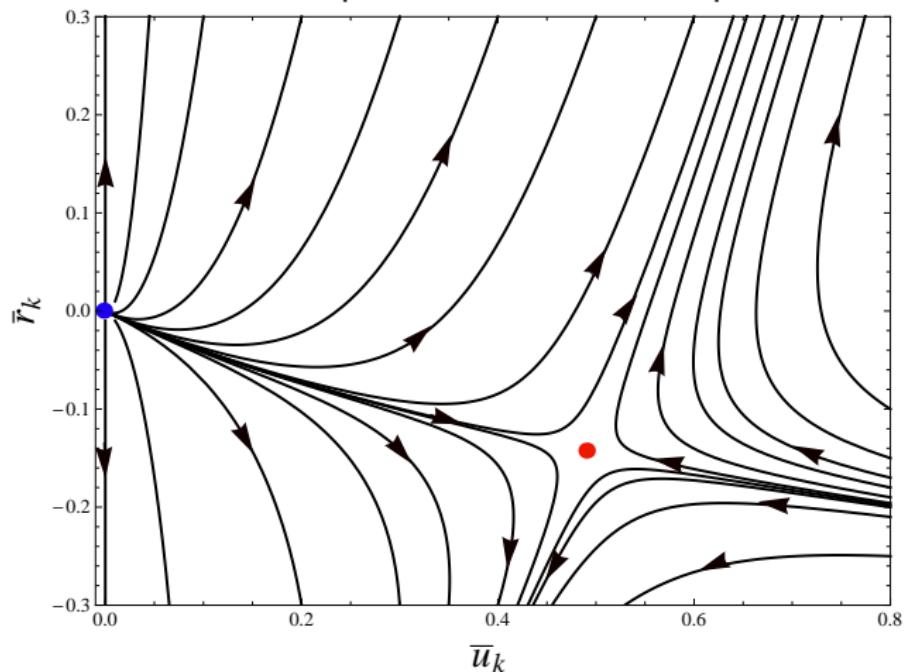
- ① the spatial dimensionality D of the system
- ② the number of components of the order parameter
- ③ the symmetry properties of the order parameter
- ④ the range and angular dependence of the interaction

The universality hypothesis

Wilson: stable IR fixed point $\Rightarrow \exists$ 2nd order phase transition

The universality hypothesis

Wilson: stable IR fixed point $\Rightarrow \exists$ 2nd order phase transition



Universality class of two-flavor QCD

$$U(N_f = 2)_L \times U(N_f = 2)_R$$

Universality class of two-flavor QCD

$$U(N_f = 2)_L \times U(N_f = 2)_R \simeq U(1)_A \times SU(2)_A \times SU(2)_V \times U(1)_V$$

Most general renormalizable Lagrangian for a complex 2×2 matrix field Φ (i.e. 8 d.o.f.) invariant under $U(2)_A \times U(2)_V$

$$\begin{aligned}\mathcal{L} = & \frac{1}{2} \text{Tr}(\partial_\mu \Phi^\dagger)(\partial_\mu \Phi) + \frac{1}{2} m_\Phi^2 \text{Tr} \Phi^\dagger \Phi \\ & + \frac{\pi^2}{3} g_1 (\text{Tr} \Phi^\dagger \Phi)^2 + \frac{\pi^2}{3} g_2 \text{Tr} (\Phi^\dagger \Phi)^2 ,\end{aligned}$$

$$\text{where } \Phi = (\sigma + i\eta) t_0 + \vec{t} \cdot (\vec{a} + i\vec{\pi}).$$

Universality class of two-flavor QCD

$$U(N_f = 2)_L \times U(N_f = 2)_R \simeq U(1)_A \times SU(2)_A \times SU(2)_V \times U(1)_V$$

8 d.o.f.: scalar $(\sigma, a_0^+, a_0^0, a_0^-)$ and pseudoscalar $(\eta, \pi^+, \pi^-, \pi^-)$ mesons

Most general renormalizable Lagrangian for a complex 2×2 matrix field Φ (i.e. 8 d.o.f.) invariant under $U(2)_A \times U(2)_V$

$$\begin{aligned}\mathcal{L} = & \frac{1}{2} \text{Tr}(\partial_\mu \Phi^\dagger)(\partial_\mu \Phi) + \frac{1}{2} m_\Phi^2 \text{Tr} \Phi^\dagger \Phi \\ & + \frac{\pi^2}{3} g_1 (\text{Tr} \Phi^\dagger \Phi)^2 + \frac{\pi^2}{3} g_2 \text{Tr} (\Phi^\dagger \Phi)^2 ,\end{aligned}$$

$$\text{where } \Phi = (\sigma + i\eta) t_0 + \vec{t} \cdot (\vec{a} + i\vec{\pi}).$$

Universality class of two-flavor QCD

$U(N_f = 2)_L \times U(N_f = 2)_R \simeq \cancel{U(1)_A} \times SU(2)_A \times U(2)_V$
8 d.o.f.: scalar $(\sigma, a_0^+, a_0^0, a_0^-)$ and pseudoscalar $(\eta, \pi^+, \pi^-, \pi^-)$
mesons

Not most general renormalizable Lagrangian for a complex 2×2 matrix field Φ (i.e. 8 d.o.f.) invariant under $SU(2)_A \times U(2)_V$

$$\begin{aligned}\mathcal{L} = & \frac{1}{2} \text{Tr}(\partial_\mu \Phi^\dagger)(\partial_\mu \Phi) + \frac{1}{2} m_\Phi^2 \text{Tr} \Phi^\dagger \Phi \\ & + \frac{\pi^2}{3} g_1 (\text{Tr} \Phi^\dagger \Phi)^2 + \frac{\pi^2}{3} g_2 \text{Tr} (\Phi^\dagger \Phi)^2 \\ & c \left(\det \Phi^\dagger + \det \Phi \right) + y \left(\det \Phi^\dagger + \det \Phi \right) \text{Tr} \Phi^\dagger \Phi \\ & + z \left[\left(\det \Phi^\dagger \right)^2 + \left(\det \Phi \right)^2 \right],\end{aligned}$$

where $\Phi = (\sigma + i\eta) t_0 + \vec{t} \cdot (\vec{a} + i\vec{\pi})$.

The $O(4)$ -conjecture

First proposed by Pisarski and Wilczek in 1983
[**Phys.Rev.D29** 338, cited 776 times]

The $O(4)$ -conjecture

First proposed by Pisarski and Wilczek in 1983
[**Phys.Rev.D29** 338, cited 776 times]

8 d.o.f.: scalar $(\sigma, a_0^+, a_0^0, a_0^-)$ and pseudoscalar $(\eta, \pi^+, \pi^-, \pi^-)$
mesons

$$\Phi = (\sigma + i\eta) t_0 + \vec{t} \cdot (\vec{a} + i\vec{\pi})$$

The $O(4)$ -conjecture

First proposed by Pisarski and Wilczek in 1983
[Phys.Rev.D29 338, cited 776 times]

8 d.o.f.: scalar $(\sigma, a_0^+, a_0^0, a_0^-)$ and pseudoscalar $(\eta, \pi^+, \pi^-, \pi^-)$ mesons

$$\Phi = (\sigma + i\eta) t_0 + \vec{t} \cdot (\vec{a} + i\vec{\pi})$$

$$SU(2) \times SU(2)/Z(2) \simeq SO(4) \sim O(4)$$

$$\Phi_1 = \sigma t_0 + i \vec{t} \cdot \vec{\pi}, \quad \Phi_2 = i \eta t_0 + \vec{t} \cdot \vec{a}$$

FRG in Local Potential Approximation (LPA)

$$\begin{aligned}\mathcal{L} = & \frac{1}{2} \text{Tr}(\partial_\mu \Phi^\dagger)(\partial_\mu \Phi) + \frac{1}{2} m_\Phi^2 \text{Tr} \Phi^\dagger \Phi \\ & + \frac{\pi^2}{3} g_1 (\text{Tr} \Phi^\dagger \Phi)^2 + \frac{\pi^2}{3} g_2 \text{Tr} (\Phi^\dagger \Phi)^2\end{aligned}$$

FRG in Local Potential Approximation (LPA)

$$U_\Lambda = \frac{1}{2} m_\Phi^2 \text{Tr} \Phi^\dagger \Phi + \frac{\pi^2}{3} g_1 (\text{Tr} \Phi^\dagger \Phi)^2 + \frac{\pi^2}{3} g_2 \text{Tr} (\Phi^\dagger \Phi)^2$$

FRG in Local Potential Approximation (LPA)

$$U_\Lambda = \frac{1}{2} m_\Phi^2 \text{Tr} \Phi^\dagger \Phi + \frac{\pi^2}{3} g_1 (\text{Tr} \Phi^\dagger \Phi)^2 + \frac{\pi^2}{3} g_2 \text{Tr} (\Phi^\dagger \Phi)^2$$

$$U_k = \frac{1}{2} m_{\Phi,k}^2 \text{Tr} \Phi^\dagger \Phi + \frac{\pi^2}{3} g_{1,k} (\text{Tr} \Phi^\dagger \Phi)^2 + \frac{\pi^2}{3} g_{2,k} \text{Tr} (\Phi^\dagger \Phi)^2$$

FRG in Local Potential Approximation (LPA)

$$U_\Lambda = \frac{1}{2} m_\Phi^2 \text{Tr} \Phi^\dagger \Phi + \frac{\pi^2}{3} g_1 (\text{Tr} \Phi^\dagger \Phi)^2 + \frac{\pi^2}{3} g_2 \text{Tr} (\Phi^\dagger \Phi)^2$$

$$U_k = \frac{1}{2} m_{\Phi,k}^2 \text{Tr} \Phi^\dagger \Phi + \frac{\pi^2}{3} g_{1,k} (\text{Tr} \Phi^\dagger \Phi)^2 + \frac{\pi^2}{3} g_{2,k} \text{Tr} (\Phi^\dagger \Phi)^2$$

Wetterich equation, LPA with Litim regulator

$$\partial_k U_k[\phi_i] = K_d k^{d+1} \sum_i \frac{1}{E_i^2}$$

FRG in Local Potential Approximation (LPA)

$$U_\Lambda = \frac{1}{2} m_\Phi^2 \text{Tr} \Phi^\dagger \Phi + \frac{\pi^2}{3} g_1 (\text{Tr} \Phi^\dagger \Phi)^2 + \frac{\pi^2}{3} g_2 \text{Tr} (\Phi^\dagger \Phi)^2$$

$$U_k = \frac{1}{2} m_{\Phi,k}^2 \text{Tr} \Phi^\dagger \Phi + \frac{\pi^2}{3} g_{1,k} (\text{Tr} \Phi^\dagger \Phi)^2 + \frac{\pi^2}{3} g_{2,k} \text{Tr} (\Phi^\dagger \Phi)^2$$

Wetterich equation, LPA with Litim regulator

$$\partial_k U_k[\phi_i] = K_d k^{d+1} \sum_i \frac{1}{E_i^2}$$

$$E_i^2 \equiv k^2 + M_i^2 , \quad M_{ij} \equiv \frac{\partial^2 U_k}{\partial \phi_i \partial \phi_j} , \quad i,j = 1, \dots, 8$$

$U(2)_V \times U(2)_A$ linear sigma model

Pisarski, Wilczek (1983): no stable fixed point in absence of anomaly (ϵ -expansion, 1 loop)

$U(2)_V \times U(2)_A$ linear sigma model

Pisarski, Wilczek (1983): no stable fixed point in absence of anomaly (ϵ -expansion, 1 loop)

Calabrese, Parruccini **JHEP 0405 018** (2004): verification to 5 loop order

$U(2)_V \times U(2)_A$ linear sigma model

Pisarski, Wilczek (1983): no stable fixed point in absence of anomaly (ϵ -expansion, 1 loop)

Calabrese, Parruccini **JHEP 0405 018** (2004): verification to 5 loop order

Espriu, Koulovassilopoulos, Travesset

Nucl.Phys.Proc.Supp.63:572 (1998): verification from Lattice

$U(2)_V \times U(2)_A$ linear sigma model

Pisarski, Wilczek (1983): no stable fixed point in absence of anomaly (ϵ -expansion, 1 loop)

Calabrese, Parruccini **JHEP 0405 018** (2004): verification to 5 loop order

Espriu, Koulovassilopoulos, Travesset

Nucl.Phys.Proc.Supp.63:572 (1998): verification from Lattice
Fukushima, Kamikado, Klein [[arXiv:1010.6226v1](#)] (Oct.2010):
verification from FRG (LPA)

(earlier: Berges, Tetradis, Wetterich **Phys.Rep.363:223** (2002))

$U(2)_V \times U(2)_A$ linear sigma model

Pisarski, Wilczek (1983): no stable fixed point in absence of anomaly (ϵ -expansion, 1 loop)

Calabrese, Parruccini **JHEP 0405 018** (2004): verification to 5 loop order

Espriu, Koulovassilopoulos, Travesset

Nucl.Phys.Proc.Supp.63:572 (1998): verification from Lattice
Fukushima, Kamikado, Klein [[arXiv:1010.6226v1](#)] (Oct.2010):
verification from FRG (LPA)

(earlier: Berges, Tetradis, Wetterich **Phys.Rep.363:223** (2002))

New:

$$c (\det \Phi^\dagger + \det \Phi) + y (\det \Phi^\dagger + \det \Phi) \operatorname{Tr} \Phi^\dagger \Phi + z \left[(\det \Phi^\dagger)^2 + (\det \Phi)^2 \right]$$

8 d.o.f.

$U(2)_V \times SU(2)_A$ linear sigma model

$$\begin{aligned}\mathcal{L}_\Phi = & \frac{1}{2} \text{Tr}(\partial_\mu \Phi^\dagger)(\partial_\mu \Phi) + \frac{1}{2} m_\Phi^2 \text{Tr} \Phi^\dagger \Phi \\ & + \frac{\pi^2}{3} g_1 (\text{Tr} \Phi^\dagger \Phi)^2 + \frac{\pi^2}{3} g_2 \text{Tr} (\Phi^\dagger \Phi)^2 \\ & + c \left(\det \Phi^\dagger + \det \Phi \right) + y \left(\det \Phi^\dagger + \det \Phi \right) \text{Tr} \Phi^\dagger \Phi \\ & + z \left[\left(\det \Phi^\dagger \right)^2 + \left(\det \Phi \right)^2 \right]\end{aligned}$$

$U(2)_V \times SU(2)_A$ linear sigma model

$$\begin{aligned}\mathcal{L}_\Phi = & \frac{1}{2} \text{Tr}(\partial_\mu \Phi^\dagger)(\partial_\mu \Phi) + \frac{1}{2} m_\Phi^2 \text{Tr} \Phi^\dagger \Phi \\ & + \frac{\pi^2}{3} g_1 (\text{Tr} \Phi^\dagger \Phi)^2 + \frac{\pi^2}{3} g_2 \text{Tr} (\Phi^\dagger \Phi)^2 \\ & + c \left(\det \Phi^\dagger + \det \Phi \right) + y \left(\det \Phi^\dagger + \det \Phi \right) \text{Tr} \Phi^\dagger \Phi \\ & + z \left[\left(\det \Phi^\dagger \right)^2 + \left(\det \Phi \right)^2 \right]\end{aligned}$$

$$\begin{aligned}\lambda_1 &\equiv 4! \frac{\pi^2}{3} \left(g_1 + \frac{1}{2} g_2 \right) , \quad \lambda_2 \equiv 2 \frac{\pi^2}{3} g_2 , \quad \mu^2 \equiv m_\Phi^2 \\ \Phi &= (\sigma + i\eta) t_0 + \vec{t} \cdot (\vec{a} + i\vec{\pi})\end{aligned}$$

$U(2)_V \times SU(2)_A$ linear sigma model

$$U(\varphi, \xi, \alpha) = \frac{1}{2}\mu^2\varphi + \frac{1}{4!}\lambda_1\varphi^2 + \lambda_2\xi + c\alpha + y\alpha\varphi + z\beta$$

$$\varphi \equiv \sigma^2 + \vec{\pi}^2 + \eta^2 + \vec{a}^2 , \quad \xi = (\sigma^2 + \vec{\pi}^2)(\eta^2 + \vec{a}^2) - (\sigma\eta - \vec{\pi} \cdot \vec{a})^2 ,$$

$$\alpha \equiv \sigma^2 - \eta^2 + \vec{\pi}^2 - \vec{a}^2 ,$$

$$\begin{aligned} \beta \equiv & \frac{1}{2} (\eta^2 + \vec{a}^2 - \sigma^2 - \vec{\pi}^2 - 2\vec{a} \cdot \vec{\pi} + 2\eta\sigma) \times \\ & \times (\eta^2 + \vec{a}^2 - \sigma^2 - \vec{\pi}^2 + 2\vec{a} \cdot \vec{\pi} - 2\eta\sigma) . \end{aligned}$$

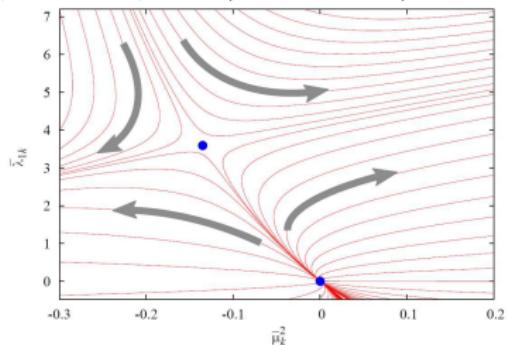
Results $c = y = z = 0$

$$\varphi \equiv \sigma^2 + \vec{\pi}^2 + \eta^2 + \vec{a}^2 , \quad \xi = (\sigma^2 + \vec{\pi}^2)(\eta^2 + \vec{a}^2) - (\sigma\eta - \vec{\pi} \cdot \vec{a})^2$$

$$U(\varphi, \xi, \alpha) = \frac{1}{2}\mu^2\varphi + \frac{1}{4!}\lambda_1\varphi^2 + \lambda_2\xi$$

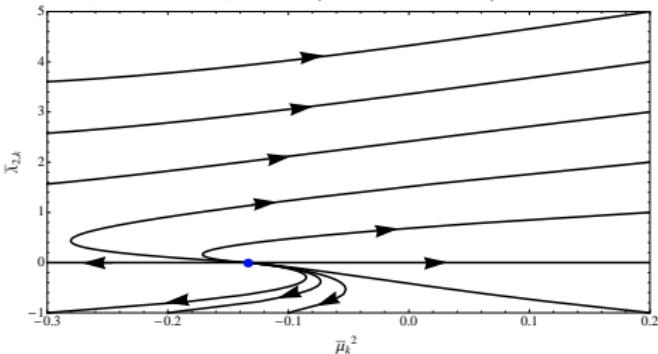
$$\partial_k U_k[\phi_i] \Big|_{\sigma, a^1 \neq 0} = K_d k^{d+1} \sum_i \frac{1}{E_i^2} \Big|_{\sigma, a^1 \neq 0}$$

$\bar{\mu}_k^2 = k^{-2}\mu_k^2$, $\bar{\lambda}_{1,k} = k^{d-4}\lambda_{1,k}$ plane:



[Fukushima et al.]

$\bar{\mu}_k^2 = k^{-2}\mu_k^2$, $\bar{\lambda}_{2,k} = k^{d-4}\lambda_{2,k}$ plane:



unstable plane

Results $c = y = z = 0$

$$(S_{ij}) \equiv \left(\frac{\partial(k\partial_k \bar{p}_i)}{\partial \bar{p}_j} \right) \Big|_{\bar{p}=\bar{p}^*}$$
$$\xi \propto |T - T_c|^{-\nu}$$

$$FP_0 \equiv (\bar{\mu}_*^2, \bar{\lambda}_{1*}, \bar{\lambda}_{2*}) = (-0.135, 3.591, 0)$$

stability matrix eigenvalues: $\{-1.71971, 1.34471, -0.25\}$

Results $c = y = z = 0$

$$(S_{ij}) \equiv \left(\frac{\partial(k\partial_k \bar{p}_i)}{\partial \bar{p}_j} \right) \Big|_{\bar{p}=\bar{p}^*}$$

$$\xi \propto |T - T_c|^{-\nu}$$

$$FP_0 \equiv (\bar{\mu}_*^2, \bar{\lambda}_{1*}, \bar{\lambda}_{2*}) = (-0.135, 3.591, 0)$$

stability matrix eigenvalues: { -1.71971, 1.34471, -0.25 }

$$U = \frac{1}{2}\mu^2 \sum_{n=1}^N \phi_i^2 + \frac{\lambda_1}{24} \left(\sum_{n=1}^N \phi_i^2 \right)^2$$

N	$\frac{1}{2}\bar{\mu}_*^2$	$\bar{\lambda}_{1*}$	$\nu = -1/y_1$	y_2
1	-0.03846	7.76271	0.54272=-1/-1.84256	1.1759
2	-0.04545	6.67366	0.55149=-1/-1.81327	1.21327
4	-0.05556	5.1988	0.564751=-1/-1.77069	1.27069
8	-0.06757	3.59143	0.581495=-1/-1.71971	1.34471

Results $c = y = z = 0$

$$(S_{ij}) \equiv \left(\frac{\partial(k\partial_k \bar{p}_i)}{\partial \bar{p}_j} \right) \Big|_{\bar{p}=\bar{p}^*}$$

$$\xi \propto |T - T_c|^{-\nu}$$

$$FP_0 \equiv (\bar{\mu}_*^2, \bar{\lambda}_{1*}, \bar{\lambda}_{2*}) = (-0.135, 3.591, 0)$$

stability matrix eigenvalues: { **-1.71971, 1.34471**, -0.25 }

$$U = \frac{1}{2}\mu^2 \sum_{n=1}^N \phi_i^2 + \frac{\lambda_1}{24} \left(\sum_{n=1}^N \phi_i^2 \right)^2$$

N	$\frac{1}{2}\bar{\mu}_*^2$	$\bar{\lambda}_{1*}$	$\nu = -1/y_1$	y_2
1	-0.03846	7.76271	0.54272=-1/-1.84256	1.1759
2	-0.04545	6.67366	0.55149=-1/-1.81327	1.21327
4	-0.05556	5.1988	0.564751=-1/-1.77069	1.27069
8	-0.06757	3.59143	0.581495=-1/ -1.71971	1.34471

Results $c = y = z = 0$

At the $O(8)$ fixed point:

	$O(8)$ model	$U(2) \times U(2)$ model
\bar{M}_σ^2	0.27027	0.27027
$\bar{M}_i^2, i \neq \sigma$	0	0
$\bar{\sigma}_0$	0.47515	0.47515
$\bar{U}(\bar{\sigma}, \vec{0})$	$-0.06757 \bar{\sigma}^2 + 0.14964 \bar{\sigma}^4$	$-0.06757 \bar{\sigma}^2 + 0.14964 \bar{\sigma}^4$

Results $c \neq 0, y \neq 0, z = 0$

$$U(\varphi, \xi, \alpha) = \frac{1}{2}\mu^2\varphi + \frac{1}{4!}\lambda_1\varphi^2 + \lambda_2\xi + c\alpha + y\alpha\varphi$$

$$\partial_k U_k[\phi_i] \Big|_{\sigma,a^1 \neq 0} = K_d k^{d+1} \sum_i \frac{1}{E_i^2} \Big|_{\sigma,a^1 \neq 0}$$

Results $c \neq 0$, $y \neq 0$, $z = 0$

$$U(\varphi, \xi, \alpha) = \frac{1}{2}\mu^2\varphi + \frac{1}{4!}\lambda_1\varphi^2 + \lambda_2\xi + c\alpha + y\alpha\varphi$$

$$\partial_k U_k[\phi_i] \Big|_{\sigma, a^1 \neq 0} = K_d k^{d+1} \sum_i \frac{1}{E_i^2} \Big|_{\sigma, a^1 \neq 0}$$

$$U = m_1^2\sigma^2 + m_2^2(a^1)^2 + \delta\sigma^2(a^1)^2 + \lambda_\sigma\sigma^4 + \lambda_a(a^1)^4$$

$$m_1^2 \equiv \frac{1}{2}\mu^2 + c, \quad m_2^2 = \frac{1}{2}\mu^2 - c,$$

$$\delta \equiv \frac{1}{12}\lambda_1 + \lambda_2, \quad \lambda_\sigma \equiv \frac{\lambda_1}{24} + y, \quad \lambda_a \equiv \frac{\lambda_1}{24} - y.$$

Results $c \neq 0, y \neq 0, z = 0$

$$U(\varphi, \xi, \alpha) = \frac{1}{2}\mu^2\varphi + \frac{1}{4!}\lambda_1\varphi^2 + \lambda_2\xi + c\alpha + y\alpha\varphi$$

$$\partial_k U_k[\phi_i] \Big|_{\sigma, a^1 \neq 0} = K_d k^{d+1} \sum_i \frac{1}{E_i^2} \Big|_{\sigma, a^1 \neq 0}$$

$$U = m_1^2\sigma^2 + m_2^2(a^1)^2 + \delta\sigma^2(a^1)^2 + \lambda_\sigma\sigma^4 + \lambda_a(a^1)^4$$

$$m_1^2 \equiv \frac{1}{2}\mu^2 + c, \quad m_2^2 = \frac{1}{2}\mu^2 - c,$$

$$\delta \equiv \frac{1}{12}\lambda_1 + \lambda_2, \quad \lambda_\sigma \equiv \frac{\lambda_1}{24} + y, \quad \lambda_a \equiv \frac{\lambda_1}{24} - y.$$

Results $c \neq 0, y \neq 0, z = 0$

$$U(\varphi, \xi, \alpha) = \frac{1}{2}\mu^2\varphi + \frac{1}{4!}\lambda_1\varphi^2 + \lambda_2\xi + c\alpha + y\alpha\varphi$$

$$\partial_k U_k[\phi_i] \Big|_{\sigma, a^1 \neq 0} = K_d k^{d+1} \sum_i \frac{1}{E_i^2} \Big|_{\sigma, a^1 \neq 0}$$

$$U = m_1^2\sigma^2 + m_2^2(a^1)^2 + \delta\sigma^2(a^1)^2 + \lambda_\sigma\sigma^4 + \lambda_a(a^1)^4$$

$$m_1^2 \equiv \frac{1}{2}\mu^2 + c, \quad m_2^2 = \frac{1}{2}\mu^2 - c,$$

$$\delta \equiv \frac{1}{12}\lambda_1 + \lambda_2, \quad \lambda_\sigma \equiv \frac{\lambda_1}{24} + y, \quad \lambda_a \equiv \frac{\lambda_1}{24} - y.$$

Ambiguous flow equations for $c \neq 0, y = 0, z = 0$

Our results for $c \neq 0, y = 0, z = 0$ assume $\sigma \gg a^1$

Results $c \neq 0, y \neq 0, z = 0$

$$\bar{m}_{i,k}^2 \equiv k^{-2} m_{i,k}^2 \quad \text{etc.}$$

$$A \equiv (\bar{m}_{1*}^2, \bar{m}_{2*}^2, \bar{\delta}_*, \bar{\lambda}_{\sigma*}, \bar{\lambda}_{a*}) = (-0.0599, -0.0083, 0.0406, 0.2067, 0.0050)$$

stability matrix eigenvalues: $\{-1.99923, -1.78902, 1.28814, -0.942028, -0.57213\}$

$$A' \equiv (\bar{m}_{1*}^2, \bar{m}_{2*}^2, \bar{\delta}_*, \bar{\lambda}_{\sigma*}, \bar{\lambda}_{a*}) = (-0.0083, -0.0599, 0.0406, 0.0050, 0.2067)$$

stability matrix eigenvalues: $\{-1.99923, -1.78902, 1.28814, -0.942028, -0.57213\}$

$$FP_0 \equiv (\bar{m}_{1*}^2, \bar{m}_{2*}^2, \bar{\delta}_*, \bar{\lambda}_{\sigma*}, \bar{\lambda}_{a*}) = (-0.0676, -0.0676, 0.2993, 0.1496, 0.1496)$$

stability matrix eigenvalues: $\{-1.98804, -1.71971, 1.34471, 0.613041, -0.25\}$

$$B \equiv (\bar{m}_{1*}^2, \bar{m}_{2*}^2, \bar{\delta}_*, \bar{\lambda}_{\sigma*}, \bar{\lambda}_{a*}) = (0.4343, -0.8398, 22.4235, -31.1684, 1.4998)$$

stability matrix eigenvalues:

$$\{-47.1087, -20.5867, -0.5545 + 8.2796i, -0.5545 - 8.2796i, -2.27794\}$$

$$B' \equiv (\bar{m}_{1*}^2, \bar{m}_{2*}^2, \bar{\delta}_*, \bar{\lambda}_{\sigma*}, \bar{\lambda}_{a*}) = (-0.8398, 0.4343, 22.4235, 1.4998, -31.1684)$$

stability matrix eigenvalues:

$$\{-47.1087, -20.5867, -0.5545 + 8.2796i, -0.5545 - 8.2796i, -2.27794\}$$

Results $c \neq 0, y \neq 0, z = 0$

$$\bar{m}_{i,k}^2 \equiv k^{-2} m_{i,k}^2 \quad \text{etc.}$$

$$A \equiv (\bar{m}_{1*}^2, \bar{m}_{2*}^2, \bar{\delta}_*, \bar{\lambda}_{\sigma*}, \bar{\lambda}_{a*}) = (-0.0599, -0.0083, 0.0406, 0.2067, 0.0050)$$

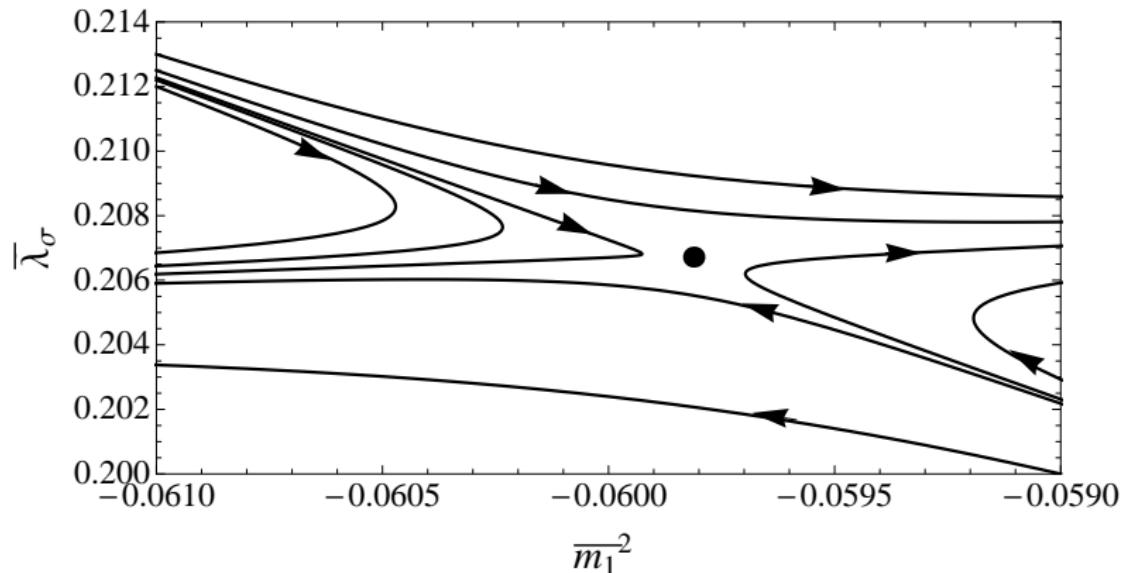
stability matrix eigenvalues: $\{-1.99923, -1.78902, 1.28814, -0.942028, -0.57213\}$

$$A' \equiv (\bar{m}_{1*}^2, \bar{m}_{2*}^2, \bar{\delta}_*, \bar{\lambda}_{\sigma*}, \bar{\lambda}_{a*}) = (-0.0083, -0.0599, 0.0406, 0.0050, 0.2067)$$

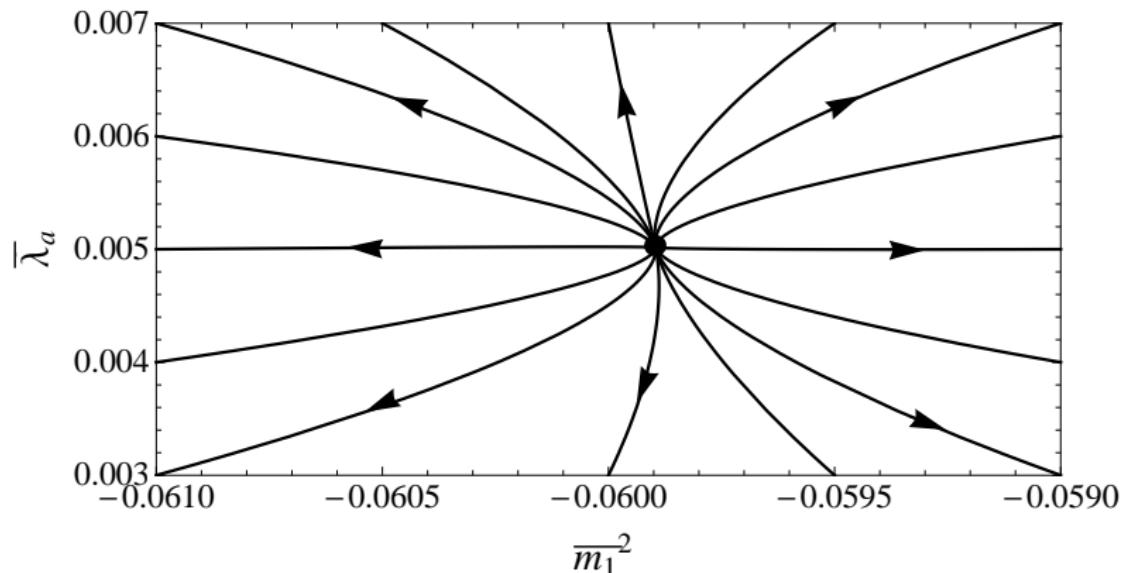
stability matrix eigenvalues: $\{-1.99923, -1.78902, 1.28814, -0.942028, -0.57213\}$

$O(N)$	$\frac{1}{2}\bar{\mu}_*^2$	$\bar{\lambda}_{1*}$	$\nu = -1/y_1$	y_2
1	-0.03846	7.76271	0.54272=-1/-1.84256	1.1759
2	-0.04545	6.67366	0.55149=-1/-1.81327	1.21327
4	-0.05556	5.1988	0.564751=-1/-1.77069	1.27069
8	-0.06757	3.59143	0.581495=-1/-1.71971	1.34471

Results $c \neq 0, y \neq 0, z = 0$, Fixed Point A



Results $c \neq 0, y \neq 0, z = 0$, Fixed Point A



Results $c \neq 0, y = 0, z = 0$

$$FP_0 \equiv (\bar{\mu}_*^2, \bar{\lambda}_{1*}, \bar{\lambda}_{2*}, \bar{c}_*) = (-0.135, 3.591, 0, 0)$$

stability matrix eigenvalues: $\{-1.71971, 1.34471, -0.25, -1.875\}$

$$FP_1 \equiv (\bar{\mu}_*^2, \bar{\lambda}_{1*}, \bar{\lambda}_{2*}, \bar{c}_*) = (-0.8512, 6.0564, -0.5797, 0.21699)$$

stability matrix eigenvalues: $\{-23.4681, -15.048, 6.752, -1.66877\}$

$$FP_2 \equiv (\bar{\mu}_*^2, \bar{\lambda}_{1*}, \bar{\lambda}_{2*}, \bar{c}_*) = (-1.1683, 1.8341, 0.8484, 0.21675)$$

stability matrix eigenvalues:

$$\{29.4962, -2.7835 + 9.9434i, -2.7835 - 9.9434i, -1.06612\}$$

Results $c \neq 0, y = 0, z = 0$

$$FP_0 \equiv (\bar{\mu}_*^2, \bar{\lambda}_{1*}, \bar{\lambda}_{2*}, \bar{c}_*) = (-0.135, 3.591, 0, 0)$$

stability matrix eigenvalues: $\{-1.71971, 1.34471, -0.25, -1.875\}$

$$FP_1 \equiv (\bar{\mu}_*^2, \bar{\lambda}_{1*}, \bar{\lambda}_{2*}, \bar{c}_*) = (-0.8512, 6.0564, -0.5797, 0.21699)$$

stability matrix eigenvalues: $\{-23.4681, -15.048, 6.752, -1.66877\}$

$$FP_2 \equiv (\bar{\mu}_*^2, \bar{\lambda}_{1*}, \bar{\lambda}_{2*}, \bar{c}_*) = (-1.1683, 1.8341, 0.8484, 0.21675)$$

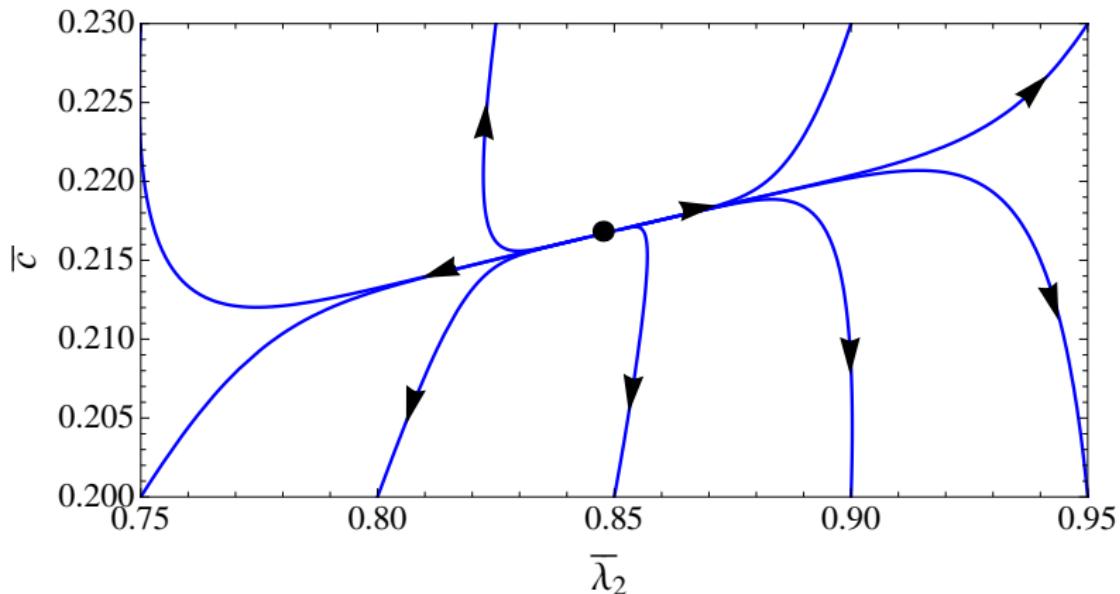
stability matrix eigenvalues:

$$\{29.4962, -2.7835 + 9.9434i, -2.7835 - 9.9434i, -1.06612\}$$

$O(N)$	$\frac{1}{2}\bar{\mu}_*^2$	$\bar{\lambda}_{1*}$	$\nu = -1/y_1$	y_2
1	-0.03846	7.76271	0.54272=-1/-1.84256	1.1759
2	-0.04545	6.67366	0.55149=-1/-1.81327	1.21327
4	-0.05556	5.1988	0.564751=-1/-1.77069	1.27069
8	-0.06757	3.59143	0.581495=-1/-1.71971	1.34471

Results $c \neq 0$, $y = 0$, $z = 0$, Fixed Point FP_2

The only unstable plane:



Results $c \neq 0, y \neq 0, z \neq 0$

$$U(\varphi, \xi, \alpha) = \frac{1}{2}\mu^2\varphi + \frac{1}{4!}\lambda_1\varphi^2 + \lambda_2\xi + c\alpha + y\alpha\varphi + z\beta$$

$$\partial_k U_k[\phi_i] \Big|_{\sigma, a_1, \eta, \pi_1 \neq 0} = K_d k^{d+1} \sum_i \frac{1}{E_i^2} \Big|_{\sigma, a_1, \eta, \pi_1 \neq 0}$$

Results $c \neq 0, y \neq 0, z \neq 0$

$$U(\varphi, \xi, \alpha) = \frac{1}{2}\mu^2\varphi + \frac{1}{4!}\lambda_1\varphi^2 + \lambda_2\xi + c\alpha + y\alpha\varphi + z\beta$$

$$\partial_k U_k[\phi_i] \Big|_{\sigma, a_1, \eta, \pi_1 \neq 0} = K_d k^{d+1} \sum_i \frac{1}{E_i^2} \Big|_{\sigma, a_1, \eta, \pi_1 \neq 0}$$

$$\begin{aligned} U_k = & a_1^2 m_{2,k}^2 + \eta^2 m_{2,k}^2 + \sigma^2 m_{1,k}^2 + \pi_1^2 m_{1,k}^2 + \left(a_1^4 + \eta^4 \right) \lambda_{a\eta} + \left(\sigma^4 + \pi_1^4 \right) \lambda_{\sigma\pi} \\ & + \delta_1 \left(\pi_1^2 a_1^2 + \eta^2 \sigma^2 \right) + \delta_2 a_1^2 \eta^2 + \delta_0 \left(a_1^2 \sigma^2 + \pi_1^2 \eta^2 \right) + \kappa \pi_1 a_1 \eta \sigma + \delta_3 \pi_1^2 \sigma^2 , \end{aligned}$$

$$\lambda_{a\eta} \equiv \left(\frac{\lambda_1}{24} - y + \frac{z}{2} \right) , \quad \lambda_{\sigma\pi} \equiv \left(\frac{\lambda_1}{24} + y + \frac{z}{2} \right) , \quad \delta_0 \equiv \left(\frac{\lambda_1}{12} + \lambda_2 - z \right) ,$$

$$\delta_1 \equiv \left(\frac{\lambda_1}{12} - 3z \right) , \quad \delta_2 \equiv \left(\frac{\lambda_1}{12} + z - 2y \right) , \quad \delta_3 \equiv \left(\frac{\lambda_1}{12} + z + 2y \right) , \quad \kappa \equiv 4z + 2\lambda_2 .$$

Note that

$$\delta_3 = 2\lambda_{\sigma\pi} , \quad \delta_2 = 2\lambda_{a\eta} , \quad \delta_0 = \delta_1 + \frac{\kappa}{2} , \quad y = \frac{\lambda_{\sigma\pi}}{2} - \frac{\lambda_{a\eta}}{2} ,$$

$$z = -\frac{\delta_1}{4} + \frac{\lambda_{\sigma\pi}}{4} + \frac{\lambda_{a\eta}}{4} , \quad \lambda_1 = 3\delta_1 + 9\lambda_{a\eta} + 9\lambda_{\sigma\pi} , \quad \lambda_2 = \frac{\delta_1}{2} + \frac{\kappa}{2} - \frac{\lambda_{\sigma\pi}}{2} - \frac{\lambda_{a\eta}}{2} .$$

Results $c \neq 0, y \neq 0, z \neq 0$

$$F_1 \equiv (\bar{m}_{1*}^2, \bar{m}_{2*}^2, \bar{\lambda}_{1*}, \bar{\lambda}_{2*}, \bar{y}_*, \bar{z}_*) = (-0.05556, -0.05556, 3.89910, -0.21662, 0, -0.10831)$$

stability matrix eigenvalues: $\{-2, -1.77069, 1.27069, -0.66667, 0, 0\}$

$$F_3 \equiv (\bar{m}_{1*}^2, \bar{m}_{2*}^2, \bar{\lambda}_{1*}, \bar{\lambda}_{2*}, \bar{y}_*, \bar{z}_*) = (-0.05556, 0, 1.94955, -0.10831, 0.10831, 0.05415)$$

stability matrix eigenvalues: $\{-2, -1.77069, 1.27069, -1, -0.83333, -0.5\}$

$$F_5 \equiv (\bar{m}_{1*}^2, \bar{m}_{2*}^2, \bar{\lambda}_{1*}, \bar{\lambda}_{2*}, \bar{y}_*, \bar{z}_*) = (-0.05556, -0.05556, 3.89910, -0.21662, 0, 0.10831)$$

stability matrix eigenvalues: $\{-1.77069, -1.77069, 1.27069, 1.27069, -0.66667, 0\}$

$$FP_0 \equiv (\bar{m}_{1*}^2, \bar{m}_{2*}^2, \bar{\lambda}_{1*}, \bar{\lambda}_{2*}, \bar{y}_*, \bar{z}_*) = (-0.06757, -0.06757, 3.59143, 0, 0, 0)$$

stability matrix eigenvalues: $\{-1.98804, -1.71971, 1.34471, 0.61304, -0.25000, -0.25000\}$

$$F_{IRS} \equiv (\bar{m}_{1*}^2, \bar{m}_{2*}^2, \bar{\lambda}_{1*}, \bar{\lambda}_{2*}, \bar{y}_*, \bar{z}_*) = (-0.334, -1.347, -152.996, 14.242, 8.581, -4.155)$$

stability matrix eigenvalues:

$$\{29.6235, 14.1 + 4.2i, 14.1 - 4.2i, 0.9 + 9.6i, 0.9 - 9.6i, -1.17232\}$$

(other fixed points not relevant for discussion)

Results $c \neq 0, y \neq 0, z \neq 0$

$$F_1 \equiv (\bar{m}_{1*}^2, \bar{m}_{2*}^2, \bar{\lambda}_{1*}, \bar{\lambda}_{2*}, \bar{y}_*, \bar{z}_*) = (-0.05556, -0.05556, 3.89910, -0.21662, 0, -0.10831)$$

stability matrix eigenvalues: $\{-2, -1.77069, 1.27069, -0.66667, 0, 0\}$

$$F_3 \equiv (\bar{m}_{1*}^2, \bar{m}_{2*}^2, \bar{\lambda}_{1*}, \bar{\lambda}_{2*}, \bar{y}_*, \bar{z}_*) = (-0.05556, 0, 1.94955, -0.10831, 0.10831, 0.05415)$$

stability matrix eigenvalues: $\{-2, -1.77069, 1.27069, -1, -0.83333, -0.5\}$

$$F_5 \equiv (\bar{m}_{1*}^2, \bar{m}_{2*}^2, \bar{\lambda}_{1*}, \bar{\lambda}_{2*}, \bar{y}_*, \bar{z}_*) = (-0.05556, -0.05556, 3.89910, -0.21662, 0, 0.10831)$$

stability matrix eigenvalues: $\{-1.77069, -1.77069, 1.27069, 1.27069, -0.66667, 0\}$

$$FP_0 \equiv (\bar{m}_{1*}^2, \bar{m}_{2*}^2, \bar{\lambda}_{1*}, \bar{\lambda}_{2*}, \bar{y}_*, \bar{z}_*) = (-0.06757, -0.06757, 3.59143, 0, 0, 0)$$

stability matrix eigenvalues: $\{-1.98804, -1.71971, 1.34471, 0.61304, -0.25000, -0.25000\}$

$$F_{IRS} \equiv (\bar{m}_{1*}^2, \bar{m}_{2*}^2, \bar{\lambda}_{1*}, \bar{\lambda}_{2*}, \bar{y}_*, \bar{z}_*) = (-0.334, -1.347, -152.996, 14.242, 8.581, -4.155)$$

stability matrix eigenvalues:

$$\{29.6235, 14.1 + 4.2i, 14.1 - 4.2i, 0.9 + 9.6i, 0.9 - 9.6i, -1.17232\}$$

IR stable, BUT: one (rescaled) mass matrix eigenvalue negative \Rightarrow reject F_{IRS}

(other fixed points not relevant for discussion)

Results $c \neq 0, y \neq 0, z \neq 0$

$$F_1 \equiv (\bar{m}_{1*}^2, \bar{m}_{2*}^2, \bar{\lambda}_{1*}, \bar{\lambda}_{2*}, \bar{y}_*, \bar{z}_*) = (-0.05556, -0.05556, 3.89910, -0.21662, 0, -0.10831)$$

stability matrix eigenvalues: $\{-2, -1.77069, 1.27069, -0.66667, 0, 0\}$

$$F_3 \equiv (\bar{m}_{1*}^2, \bar{m}_{2*}^2, \bar{\lambda}_{1*}, \bar{\lambda}_{2*}, \bar{y}_*, \bar{z}_*) = (-0.05556, 0, 1.94955, -0.10831, 0.10831, 0.05415)$$

stability matrix eigenvalues: $\{-2, -1.77069, 1.27069, -1, -0.83333, -0.5\}$

$$F_5 \equiv (\bar{m}_{1*}^2, \bar{m}_{2*}^2, \bar{\lambda}_{1*}, \bar{\lambda}_{2*}, \bar{y}_*, \bar{z}_*) = (-0.05556, -0.05556, 3.89910, -0.21662, 0, 0.10831)$$

stability matrix eigenvalues: $\{-1.77069, -1.77069, 1.27069, 1.27069, -0.66667, 0\}$

$O(N)$	$\frac{1}{2}\bar{\mu}_*^2$	$\bar{\lambda}_{1*}$	$\nu = -1/y_1$	y_2
1	-0.03846	7.76271	0.54272=-1/-1.84256	1.1759
2	-0.04545	6.67366	0.55149=-1/-1.81327	1.21327
4	-0.05556	5.1988	0.564751=-1/-1.77069	1.27069
8	-0.06757	3.59143	0.581495=-1/-1.71971	1.34471

Results $c \neq 0, y \neq 0, z \neq 0$

$$F_1 \equiv (\bar{m}_{1*}^2, \bar{m}_{2*}^2, \bar{\lambda}_{1*}, \bar{\lambda}_{2*}, \bar{y}_*, \bar{z}_*) = (-0.05556, -0.05556, 3.89910, -0.21662, 0, -0.10831)$$

stability matrix eigenvalues: $\{-2, -1.77069, 1.27069, -0.66667, 0, 0\}$

$$F_3 \equiv (\bar{m}_{1*}^2, \bar{m}_{2*}^2, \bar{\lambda}_{1*}, \bar{\lambda}_{2*}, \bar{y}_*, \bar{z}_*) = (-0.05556, 0, 1.94955, -0.10831, 0.10831, 0.05415)$$

stability matrix eigenvalues: $\{-2, -1.77069, 1.27069, -1, -0.83333, -0.5\}$

$$F_5 \equiv (\bar{m}_{1*}^2, \bar{m}_{2*}^2, \bar{\lambda}_{1*}, \bar{\lambda}_{2*}, \bar{y}_*, \bar{z}_*) = (-0.05556, -0.05556, 3.89910, -0.21662, 0, 0.10831)$$

stability matrix eigenvalues: $\{-1.77069, -1.77069, 1.27069, 1.27069, -0.66667, 0\}$

$O(N)$	$\frac{1}{2}\bar{\mu}_*^2$	$\bar{\lambda}_{1*}$	$\nu = -1/y_1$	y_2
1	-0.03846	7.76271	0.54272=-1/-1.84256	1.1759
2	-0.04545	6.67366	0.55149=-1/-1.81327	1.21327
4	-0.05556	5.1988	0.564751=-1/-1.77069	1.27069
8	-0.06757	3.59143	0.581495=-1/-1.71971	1.34471

Results $c \neq 0, y \neq 0, z \neq 0$

$$F_1 \equiv (\bar{m}_{1*}^2, \bar{m}_{2*}^2, \bar{\lambda}_{1*}, \bar{\lambda}_{2*}, \bar{y}_*, \bar{z}_*) = (-0.05556, -0.05556, 3.89910, -0.21662, 0, -0.10831)$$

stability matrix eigenvalues: $\{-2, -1.77069, 1.27069, -0.66667, 0, 0\}$

Comparison with $c \neq 0, y \neq 0, z = 0$:

$$A \equiv (\bar{m}_{1*}^2, \bar{m}_{2*}^2, \bar{\delta}_*, \bar{\lambda}_{\sigma*}, \bar{\lambda}_{a*}) = (-0.0599, -0.0083, 0.0406, 0.2067, 0.0050)$$

stability matrix eigenvalues: $\{-1.99923, -1.78902, 1.28814, -0.942028, -0.57213\}$

$$A' \equiv (\bar{m}_{1*}^2, \bar{m}_{2*}^2, \bar{\delta}_*, \bar{\lambda}_{\sigma*}, \bar{\lambda}_{a*}) = (-0.0083, -0.0599, 0.0406, 0.0050, 0.2067)$$

stability matrix eigenvalues: $\{-1.99923, -1.78902, 1.28814, -0.942028, -0.57213\}$

$O(N)$	$\frac{1}{2}\bar{\mu}_*^2$	$\bar{\lambda}_{1*}$	$\nu = -1/y_1$	y_2
1	-0.03846	7.76271	0.54272=-1/-1.84256	1.1759
2	-0.04545	6.67366	0.55149=-1/-1.81327	1.21327
4	-0.05556	5.1988	0.564751=-1/-1.77069	1.27069
8	-0.06757	3.59143	0.581495=-1/-1.71971	1.34471

Results $c \neq 0, y \neq 0, z \neq 0$

$$F_3 \equiv (\bar{m}_{1*}^2, \bar{m}_{2*}^2, \bar{\lambda}_{1*}, \bar{\lambda}_{2*}, \bar{y}_*, \bar{z}_*) = (-0.05556, 0, 1.94955, -0.10831, 0.10831, 0.05415)$$

stability matrix eigenvalues: $\{-2, -1.77069, 1.27069, -1, -0.83333, -0.5\}$

	$O(4)$ model	$SU(2) \times U(2)$ model
\bar{M}_σ^2	0.2	0.2
$\bar{M}_i^2, i \neq \sigma$	0	0
$\langle \bar{\sigma} \rangle$	0.358099	0.358099
$\bar{U}(\bar{\sigma}, \vec{0})$	$-0.05 \bar{\sigma}^2 + 0.216617 \bar{\sigma}^4$	$-0.05 \bar{\sigma}^2 + 0.216617 \bar{\sigma}^4$

Conclusion: unstable $O(4)$ fixed point.

Results $c \neq 0, y \neq 0, z \neq 0$

$$F_5 \equiv (\bar{m}_{1*}^2, \bar{m}_{2*}^2, \bar{\lambda}_{1*}, \bar{\lambda}_{2*}, \bar{y}_*, \bar{z}_*) = (-0.05556, -0.05556, 3.89910, -0.21662, 0, 0.10831)$$

stability matrix eigenvalues: $\{-1.77069, -1.77069, 1.27069, 1.27069, -0.66667, 0\}$

Note: $(\bar{\sigma} = 0.358099, \vec{0})$ is not a minimum (i.e. negative masses).

	$O(4)$ model	$SU(2) \times U(2)$ model
\bar{M}_σ^2	0.2	0.2
$\bar{M}_{a_1}^2$	-	0.2
$\bar{M}_i^2, i \neq \sigma, a_1$	0	0
$\langle \bar{\sigma} \rangle$	0.358099	0.358099
$\langle \bar{a}_1 \rangle$	-	0.358099
$\bar{U}(\bar{\sigma}, \bar{a}_1, \vec{0})$	$-0.05 \bar{\sigma}^2 + 0.216617 \bar{\sigma}^4$	$-0.05 (\bar{\sigma}^2 + \bar{a}_1^2) + 0.216617 (\bar{\sigma}^4 + \bar{a}_1^4)$

Conclusion:
 unstable multicritical (tricritical?) fixed point.
 σ and a_1 are two distinct $O(4)$ softmodes.

Results $c \neq 0, y \neq 0, z \neq 0$

$$F_1 \equiv (\bar{m}_{1*}^2, \bar{m}_{2*}^2, \bar{\lambda}_{1*}, \bar{\lambda}_{2*}, \bar{y}_*, \bar{z}_*) = (-0.05556, -0.05556, 3.89910, -0.21662, 0, -0.10831)$$

stability matrix eigenvalues: $\{-2, -1.77069, 1.27069, -0.66667, 0, 0\}$

	$O(4)$ model	$SU(2) \times U(2)$ model
\bar{M}_σ^2	0.2	0.2
$\bar{M}_{a_1}^2$	-	0.2
$\bar{M}_i^2, i \neq \sigma, a_1$	0	0
$<\bar{\sigma}>$	0.358099	0.358099
$<\bar{a}_1>$	-	0.358099
$\bar{U}(\bar{\sigma}, \bar{a}_1, \vec{0})$	$-0.05 \bar{\sigma}^2 + 0.216617 \bar{\sigma}^4$	$-0.05 (\bar{\sigma}^2 + \bar{a}_1^2) + \frac{0.216617}{2} (\bar{\sigma}^2 + \bar{a}_1^2)^2$

Conjecture:
 unstable multicritical (bicritical?) fixed point.
 σ and a_1 form an $O(4)$ softmode.

Strong anomaly limit $m_{2,k}^2 \rightarrow \infty$

Jungnickel and Wetterich **Phys. Rev., D53:5142** (1996): limit $c \rightarrow -\infty$ should be closer to reality.

Quark-meson model: $k \sim 600 \text{ MeV}$,

$|c| \sim (958 \text{ MeV})^2$ (for $m_\eta = 958 \text{ MeV}$, $N_f = 2$).

Strong anomaly limit $m_{2,k}^2 \rightarrow \infty$

Jungnickel and Wetterich **Phys. Rev., D53:5142** (1996): limit $c \rightarrow -\infty$ should be closer to reality.

Quark-meson model: $k \sim 600 \text{ MeV}$,

$|c| \sim (958 \text{ MeV})^2$ (for $m_\eta = 958 \text{ MeV}$, $N_f = 2$).

$$m_{2,k}^2 = \frac{1}{2}\mu_k^2 - c_k \rightarrow \infty, \text{ note that } m_{1,k}^2 \equiv \frac{1}{2}\mu_k^2 + c_k.$$

Strong anomaly limit $m_{2,k}^2 \rightarrow \infty$, $c \neq 0$, $y = 0$, $z = 0$

$$(\bar{m}_{1*}^2, \bar{\lambda}_{1*}, \bar{\lambda}_{2*}) = (-0.05556, 5.1988, -0.3713)$$

stability matrix eigenvalues: $\{-1.77069, 1.27069, -0.58333\}$.

$$M_\sigma^2 = 0.2, \quad M_{\pi_i}^2 = 0, \quad M_\eta^2 \rightarrow \infty, \quad M_{a_i}^2 \rightarrow \infty.$$

$O(N)$	$\frac{1}{2}\bar{\mu}_*^2$	$\bar{\lambda}_{1*}$	$\nu = -1/y_1$	y_2
1	-0.03846	7.76271	$0.54272 = -1/-1.84256$	1.1759
2	-0.04545	6.67366	$0.55149 = -1/-1.81327$	1.21327
4	-0.05556	5.1988	$0.564751 = -1/-1.77069$	1.27069
8	-0.06757	3.59143	$0.581495 = -1/-1.71971$	1.34471

Strong anomaly limit $m_{2,k}^2 \rightarrow \infty$, $c \neq 0$, $y = 0$, $z = 0$

$$(\bar{m}_{1*}^2, \bar{\lambda}_{1*}, \bar{\lambda}_{2*}) = (-0.05556, 5.1988, -0.3713)$$

stability matrix eigenvalues: $\{-1.77069, 1.27069, -0.58333\}$.

$$M_\sigma^2 = 0.2, \quad M_{\pi_i}^2 = 0, \quad M_\eta^2 \rightarrow \infty, \quad M_{a_i}^2 \rightarrow \infty.$$

$$\bar{U} = \bar{m}_1^2 \sigma^2 + \bar{m}_2^2 (\bar{a}^1)^2 + \left(\frac{1}{12} \bar{\lambda}_1 + \bar{\lambda}_2 \right) \bar{\sigma}^2 (\bar{a}^1)^2 + \frac{\bar{\lambda}_1}{24} \bar{\sigma}^4$$

$$\bar{m}_1^2 \equiv \frac{1}{2} \bar{\mu}^2 + \bar{c}, \quad \bar{m}_2^2 = \frac{1}{2} \bar{\mu}^2 - \bar{c} \rightarrow \infty, \quad \bar{a}_1^4 \equiv 0.$$

Strong anomaly limit $m_{2,k}^2 \rightarrow \infty$, $c \neq 0$, $y = 0$, $z = 0$

$$(\bar{m}_{1*}^2, \bar{\lambda}_{1*}, \bar{\lambda}_{2*}) = (-0.05556, 5.1988, -0.3713)$$

stability matrix eigenvalues: $\{-1.77069, 1.27069, -0.58333\}$.

$$M_\sigma^2 = 0.2, \quad M_{\pi_i}^2 = 0, \quad M_\eta^2 \rightarrow \infty, \quad M_{a_i}^2 \rightarrow \infty.$$

$$\bar{U} = \bar{m}_1^2 \sigma^2 + \bar{m}_2^2 (\bar{a}^1)^2 + \left(\frac{1}{12} \bar{\lambda}_1 + \bar{\lambda}_2 \right) \bar{\sigma}^2 (\bar{a}^1)^2 + \frac{\bar{\lambda}_1}{24} \bar{\sigma}^4$$

$$\bar{m}_1^2 \equiv \frac{1}{2} \bar{\mu}^2 + \bar{c}, \quad \bar{m}_2^2 = \frac{1}{2} \bar{\mu}^2 - \bar{c} \rightarrow \infty, \quad \bar{a}_1^4 \equiv 0.$$

$$\begin{pmatrix} \frac{\partial(k\partial_k \bar{m}_1^2)}{\partial \bar{m}_1^2} & \frac{\partial(k\partial_k \bar{m}_1^2)}{\partial \bar{\lambda}_1} \\ \frac{\partial(k\partial_k \bar{\lambda}_1)}{\partial \bar{m}_1^2} & \frac{\partial(k\partial_k \bar{\lambda}_1)}{\partial \bar{\lambda}_1} \end{pmatrix}_{\bar{m}_{1*}^2, \bar{\lambda}_{1*}}$$

Strong anomaly limit $m_{2,k}^2 \rightarrow \infty$, $c \neq 0$, $y = 0$, $z = 0$

$$(\bar{m}_{1*}^2, \bar{\lambda}_{1*}, \bar{\lambda}_{2*}) = (-0.05556, 5.1988, -0.3713)$$

stability matrix eigenvalues: $\{-1.77069, 1.27069, -0.58333\}$.

$$M_\sigma^2 = 0.2, \quad M_{\pi_i}^2 = 0, \quad M_\eta^2 \rightarrow \infty, \quad M_{a_i}^2 \rightarrow \infty.$$

$$\bar{U} = \bar{m}_1^2 \sigma^2 + \bar{m}_2^2 (\bar{a}^1)^2 + \left(\frac{1}{12} \bar{\lambda}_1 + \bar{\lambda}_2 \right) \bar{\sigma}^2 (\bar{a}^1)^2 + \frac{\bar{\lambda}_1}{24} \bar{\sigma}^4$$

$$\bar{m}_1^2 \equiv \frac{1}{2} \bar{\mu}^2 + \bar{c}, \quad \bar{m}_2^2 = \frac{1}{2} \bar{\mu}^2 - \bar{c} \rightarrow \infty, \quad \bar{a}_1^4 \equiv 0.$$

$$\begin{pmatrix} \frac{\partial(k\partial_k \bar{m}_1^2)}{\partial \bar{m}_1^2} & \frac{\partial(k\partial_k \bar{m}_1^2)}{\partial \bar{\lambda}_1} \\ \frac{\partial(k\partial_k \bar{\lambda}_1)}{\partial \bar{m}_1^2} & \frac{\partial(k\partial_k \bar{\lambda}_1)}{\partial \bar{\lambda}_1} \end{pmatrix}_{\bar{m}_{1*}^2, \bar{\lambda}_{1*}}$$

Hence: $O(4)$ fixed point has to be regarded as IR stable.

Strong anomaly limit $m_{2,k}^2 \rightarrow \infty$, $c \neq 0$, $y \neq 0$, $z \neq 0$

IR stability of the $O(4)$ fixed point also expected for

$|c|, |y|, |z| \rightarrow \infty$
keeping $m_{1,k}^2$ and $\lambda_{\sigma\pi}$ finite.

$$U_k = a_1^2 m_{2,k}^2 + \eta^2 m_{2,k}^2 + \sigma^2 \textcolor{red}{m_{1,k}^2} + \pi_1^2 m_{1,k}^2 + (a_1^4 + \eta^4) \lambda_{a\eta} + (\sigma^4 + \pi_1^4) \textcolor{red}{\lambda_{\sigma\pi}}$$
$$+ \delta_1 (\pi_1^2 a_1^2 + \eta^2 \sigma^2) + \delta_2 a_1^2 \eta^2 + \delta_0 (a_1^2 \sigma^2 + \pi_1^2 \eta^2) + \kappa \pi_1 a_1 \eta \sigma + \delta_3 \pi_1^2 \sigma^2 ,$$

$$\lambda_{a\eta} \equiv \left(\frac{\lambda_1}{24} - y + \frac{z}{2} \right) , \quad \lambda_{\sigma\pi} \equiv \left(\frac{\lambda_1}{24} + y + \frac{z}{2} \right) , \quad \delta_0 \equiv \left(\frac{\lambda_1}{12} + \lambda_2 - z \right) ,$$

$$\delta_1 \equiv \left(\frac{\lambda_1}{12} - 3z \right) , \quad \delta_2 \equiv \left(\frac{\lambda_1}{12} + z - 2y \right) , \quad \delta_3 \equiv \left(\frac{\lambda_1}{12} + z + 2y \right) , \quad \kappa \equiv 4z + 2\lambda_2 .$$

Conclusion

- $O(4)$ -conjecture: If $N_f = 2$ chiral phase transition of QCD is 2nd order (possible only in presence of anomaly), then it belongs to $O(4)$ universality class

Conclusion

- $O(4)$ -conjecture: If $N_f = 2$ chiral phase transition of QCD is 2nd order (possible only in presence of anomaly), then it belongs to $O(4)$ universality class
- Most general renormalizable Lagrangian invariant under $U(2)_A \times U(2)_V$ in presence of $SU(2)_A \times U(2)_V$ -symmetric anomaly terms

Conclusion

- $O(4)$ -conjecture: If $N_f = 2$ chiral phase transition of QCD is 2nd order (possible only in presence of anomaly), then it belongs to $O(4)$ universality class
- Most general renormalizable Lagrangian invariant under $U(2)_A \times U(2)_V$ in presence of $SU(2)_A \times U(2)_V$ -symmetric anomaly terms
- (To our knowledge) first RG-study of $N_f = 2$ linear sigma model in presence of anomaly keeping all invariants

Conclusion

- $O(4)$ -conjecture: If $N_f = 2$ chiral phase transition of QCD is 2nd order (possible only in presence of anomaly), then it belongs to $O(4)$ universality class
- Most general renormalizable Lagrangian invariant under $U(2)_A \times U(2)_V$ in presence of $SU(2)_A \times U(2)_V$ -symmetric anomaly terms
- (To our knowledge) first RG-study of $N_f = 2$ linear sigma model in presence of anomaly keeping all invariants
- $U_A(1)$ anomaly present $\Rightarrow \exists O(4)$ infrared fixed point(s)

Conclusion

- $O(4)$ -conjecture: If $N_f = 2$ chiral phase transition of QCD is 2nd order (possible only in presence of anomaly), then it belongs to $O(4)$ universality class
- Most general renormalizable Lagrangian invariant under $U(2)_A \times U(2)_V$ in presence of $SU(2)_A \times U(2)_V$ -symmetric anomaly terms
- (To our knowledge) first RG-study of $N_f = 2$ linear sigma model in presence of anomaly keeping all invariants
- $U_A(1)$ anomaly present $\Rightarrow \exists O(4)$ infrared fixed point(s)
- Finite anomaly strength: one-component softmode fixed point, two multicritical fixed points, unstable in case of all possible renormalizable 't Hooft determinant like terms

Conclusion

- $O(4)$ -conjecture: If $N_f = 2$ chiral phase transition of QCD is 2nd order (possible only in presence of anomaly), then it belongs to $O(4)$ universality class
- Most general renormalizable Lagrangian invariant under $U(2)_A \times U(2)_V$ in presence of $SU(2)_A \times U(2)_V$ -symmetric anomaly terms
- (To our knowledge) first RG-study of $N_f = 2$ linear sigma model in presence of anomaly keeping all invariants
- $U_A(1)$ anomaly present $\Rightarrow \exists O(4)$ infrared fixed point(s)
- Finite anomaly strength: one-component softmode fixed point, two multicritical fixed points, unstable in case of all possible renormalizable 't Hooft determinant like terms
- Infinite anomaly strength: stable one-component softmode fixed point

Conclusion

- $O(4)$ -conjecture: If $N_f = 2$ chiral phase transition of QCD is 2nd order (possible only in presence of anomaly), then it belongs to $O(4)$ universality class
- Most general renormalizable Lagrangian invariant under $U(2)_A \times U(2)_V$ in presence of $SU(2)_A \times U(2)_V$ -symmetric anomaly terms
- (To our knowledge) first RG-study of $N_f = 2$ linear sigma model in presence of anomaly keeping all invariants
- $U_A(1)$ anomaly present $\Rightarrow \exists O(4)$ infrared fixed point(s)
- Finite anomaly strength: one-component softmode fixed point, two multicritical fixed points, unstable in case of all possible renormalizable 't Hooft determinant like terms
- Infinite anomaly strength: stable one-component softmode fixed point
- Universality hypothesis verified instead of applying it

Conclusion

- $O(4)$ -conjecture: If $N_f = 2$ chiral phase transition of QCD is 2nd order (possible only in presence of anomaly), then it belongs to $O(4)$ universality class
- Most general renormalizable Lagrangian invariant under $U(2)_A \times U(2)_V$ in presence of $SU(2)_A \times U(2)_V$ -symmetric anomaly terms
- (To our knowledge) first RG-study of $N_f = 2$ linear sigma model in presence of anomaly keeping all invariants
- $U_A(1)$ anomaly present $\Rightarrow \exists O(4)$ infrared fixed point(s)
- Finite anomaly strength: one-component softmode fixed point, two multicritical fixed points, unstable in case of all possible renormalizable 't Hooft determinant like terms
- Infinite anomaly strength: stable one-component softmode fixed point
- Universality hypothesis verified instead of applying it
- Outlook: construct the most general potential invariant under $SU(2)_A \times U(2)_V$

Conclusion

- $O(4)$ -conjecture: If $N_f = 2$ chiral phase transition of QCD is 2nd order (possible only in presence of anomaly), then it belongs to $O(4)$ universality class
- Most general renormalizable Lagrangian invariant under $U(2)_A \times U(2)_V$ in presence of $SU(2)_A \times U(2)_V$ -symmetric anomaly terms
- (To our knowledge) first RG-study of $N_f = 2$ linear sigma model in presence of anomaly keeping all invariants
- $U_A(1)$ anomaly present $\Rightarrow \exists O(4)$ infrared fixed point(s)
- Finite anomaly strength: one-component softmode fixed point, two multicritical fixed points, unstable in case of all possible renormalizable 't Hooft determinant like terms
- Infinite anomaly strength: stable one-component softmode fixed point
- Universality hypothesis verified instead of applying it
- Outlook: construct the most general potential invariant under $SU(2)_A \times U(2)_V$
- Outlook: keep nonzero Matsubara modes (\rightarrow Mario Mitter)



Special Thanks to: BNL, R.D.Pisarski, L.Bartosch,
B.Friman, F.Giacosa, H.Gies, D.H.Rischke,
S.Schramm, J.M.Pawlowski, HGS-HIRe for FAIR.